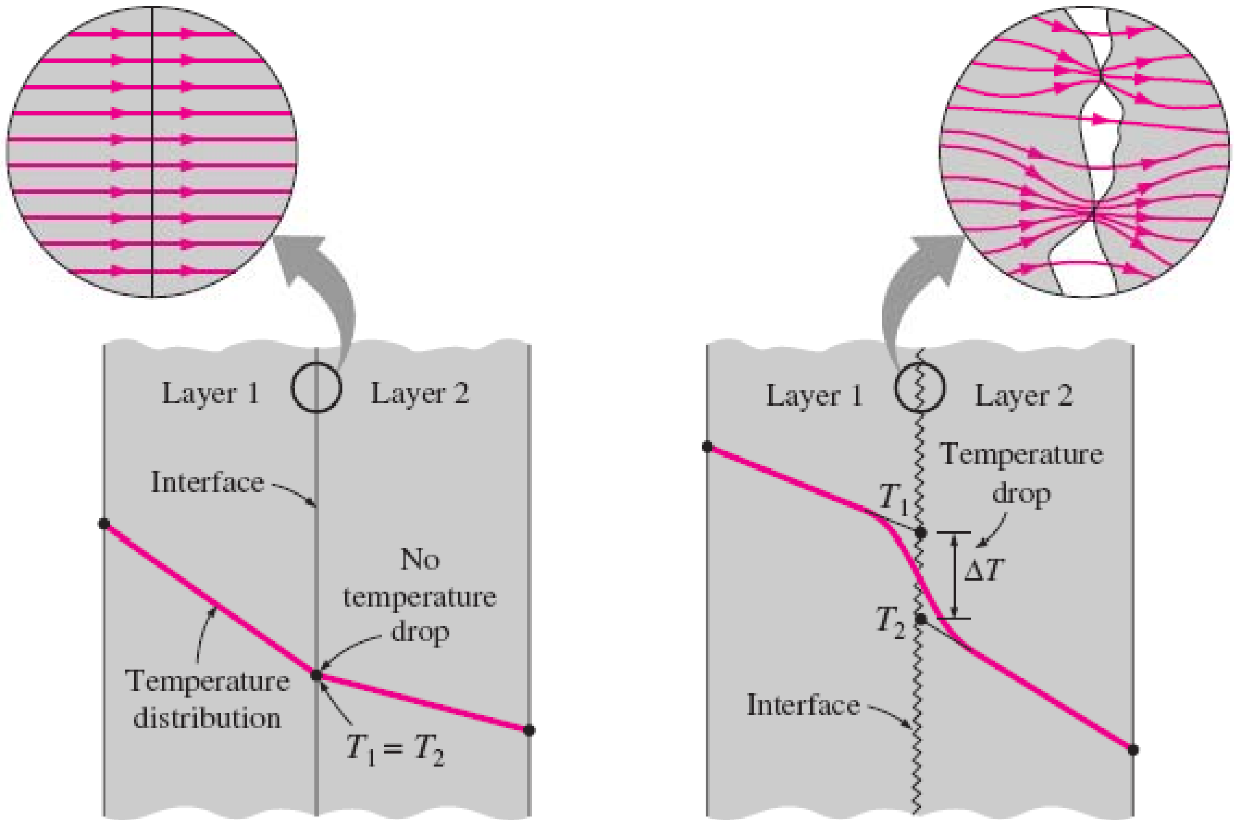
1D Steady State Heat Conduction (2)

Prabal Talukdar
Associate Professor
Department of Mechanical Engineering
IIT Delhi
E-mail: prabal@mech.iitd.ac.in

Thermal Contact Resistance



(*a*) Ideal (perfect) thermal contact

(*b*) Actual (imperfect) thermal contact

Temperature distribution and heat flow lines along two solid plates pressed against each other for the case of perfect and imperfect contact

Consider heat transfer through two metal rods of cross-sectional area A that are pressed against each other. Heat transfer through the interface of these two rods is the sum of the heat transfers through the solid contact spots and the gaps in the noncontact areas and can be expressed as

$$\dot{Q} = \dot{Q}_{contact} + \dot{Q}_{gap}$$

$$\dot{Q} = h_c A \Delta T_{interface}$$

Most experimentally determined values of the thermal contact resistance fall between 0.000005 and 0.0005 $m^2 \cdot °C/W$ (the corresponding range of thermal contact conductance is 2000 to 200,000 $W/m^2 \cdot °C$).

where A is the apparent interface area (which is the same as the cross-sectional area of the rods) and $\Delta T_{interface}$ is the effective temperature difference at the interface. The quantity h_c, which corresponds to the convection heat transfer coefficient, is called the **thermal contact conductance and is expressed as**

$$h_c = \frac{\dot{Q}/A}{\Delta T_{interface}} \qquad (W/m^2\,{}^oC)$$

It is related to thermal contact resistance by

$$R_c = \frac{1}{h_c} = \frac{\Delta T_{interface}}{\dot{Q}/A} \qquad (m^2\,{}^oC/W)$$

Thermal contact conductance for aluminum plates with different fluids at the interface for a surface roughness of 10 μm and interface pressure of 1 atm (from Fried, Ref. 5)

Fluid at the Interface	Contact Conductance, h_c, W/m^2 · °C
Air	3640
Helium	9520
Hydrogen	13,900
Silicone oil	19,000
Glycerin	37,700

Importance of consideration

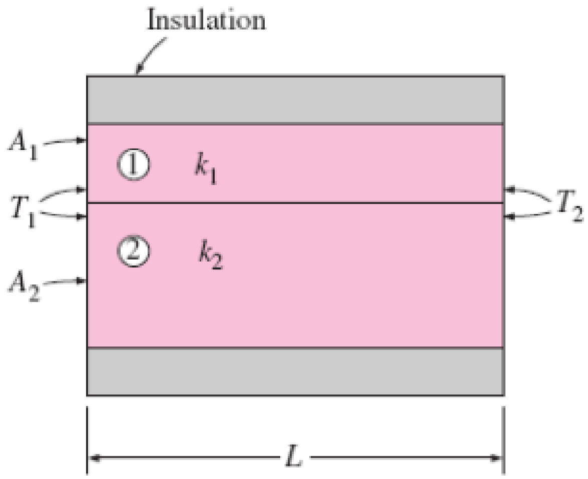
$$R_{c,\,\text{insulation}} = \frac{L}{k} = \frac{0.01\ \text{m}}{0.04\ \text{W/m}\cdot{}^\circ\text{C}} = 0.25\ \text{m}^2\cdot{}^\circ\text{C/W}$$

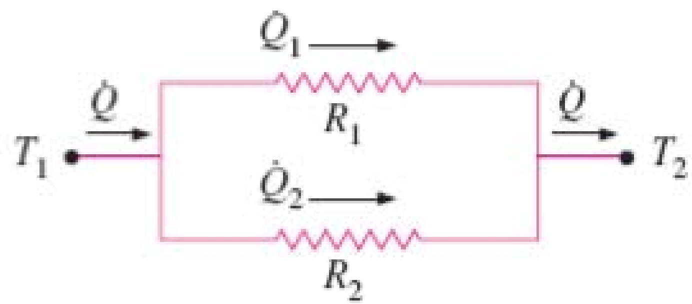
$$R_{c,\,\text{copper}} = \frac{L}{k} = \frac{0.01\ \text{m}}{386\ \text{W/m}\cdot{}^\circ\text{C}} = 0.000026\ \text{m}^2\cdot{}^\circ\text{C/W}$$

The thermal contact resistance range:

between 0.000005 and 0.0005 m^2·°C/W

Two parallel layers





$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2$$

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$R_1 = \frac{L_1}{k_1 A'_1} \qquad R_2 = \frac{L_2}{k_2 A'_2}$$

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

where $$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R_{total} = \frac{R_1 R_2}{R_1 + R_2}$$

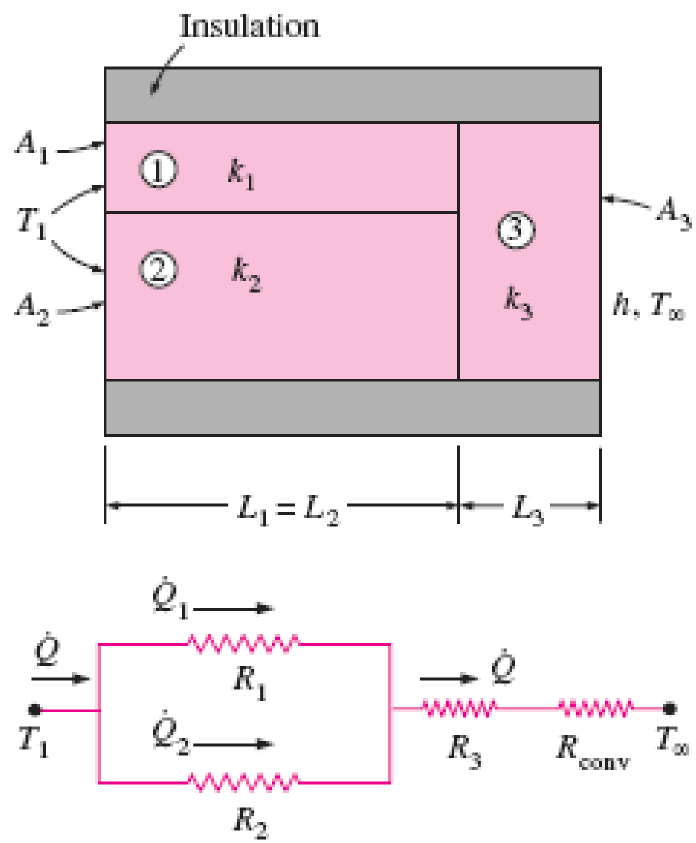
Combined series-parallel

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{total}}$$

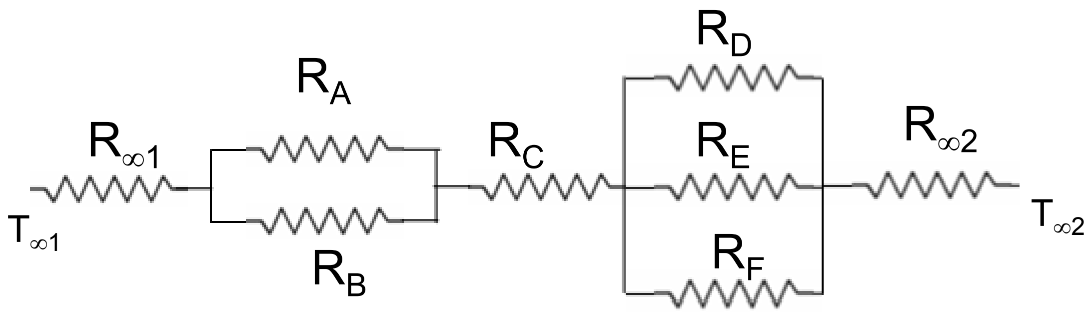
$$R_{total} = R_{12} + R_3 + R_{conv} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{conv}$$

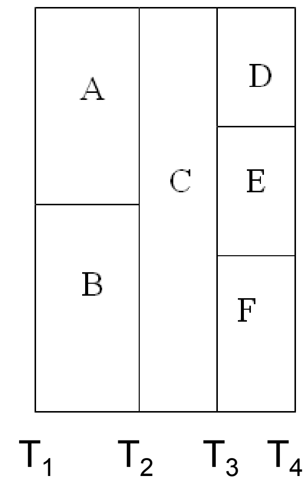
$$R_1 = \frac{L_1}{k_1 A_1} \qquad R_2 = \frac{L_2}{k_2 A_2} \qquad R_3 = \frac{L_3}{k_3 A_3}$$

$$R_{conv} = \frac{1}{hA_3}$$



Series and parallel composite wall and its thermal circuit





$$\sum R = R_{\infty 1} + \frac{1}{\left(\frac{1}{R_A} + \frac{1}{R_B}\right)} + R_C + \frac{1}{\left(\frac{1}{R_D} + \frac{1}{R_E} + \frac{1}{R_F}\right)} + R_{\infty 2}$$

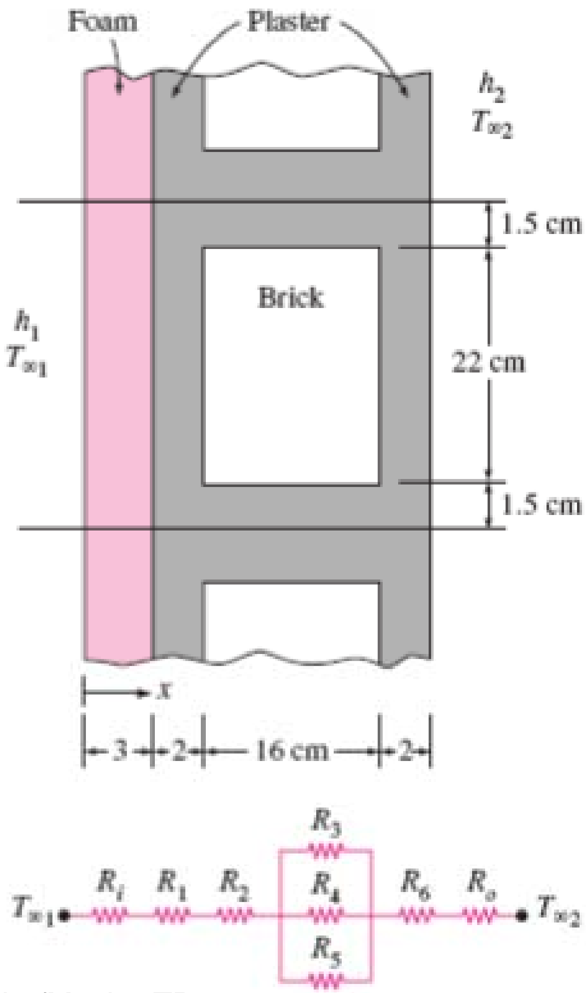
$$\dot{Q} = UA \ \Delta T \qquad \text{(W)}$$

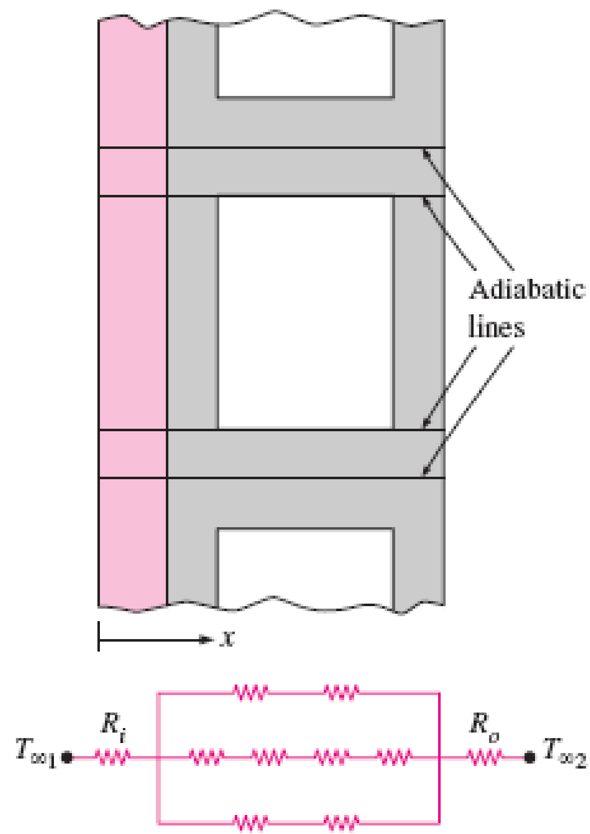
where U is the overall heat transfer coefficient

$$UA = \frac{1}{R_{total}}$$

Complex multi-dimensional problems as 1-D problems

1. Any plane wall normal to the x-axis is isothermal
2. Any plane parallel to x-axis is adiabatic





Heat conduction in cylinder

$$\dot{Q}_{cond,cyl} = -kA\frac{dT}{dr}$$

$$A = 2\pi rL$$

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{cond,cyl}}{A} dr = -\int_{T=T_1}^{T_2} kdT$$

Substituting $A = 2\pi rL$ and performing the integrations give

$$\dot{Q}_{cond,cyl} = 2\pi Lk \frac{T_1 - T_2}{\ln(r_2/r_1)}$$

$\dot{Q}_{cond,cyl}$ =constant at steady state

$$\dot{Q}_{cond,cyl} = \frac{T_1 - T_2}{R_{cyl}} \quad \text{(W)}$$

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\text{ln(outer radius/inner radius)}}{2\pi\text{(length)(thermal conductivity)}}$$

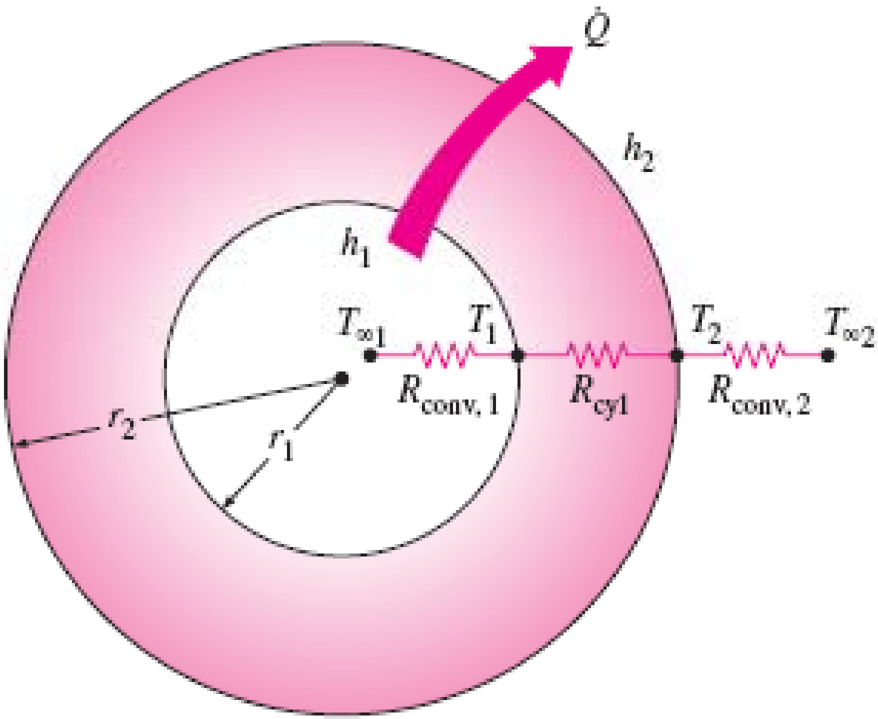
Heat conduction in sphere

For sphere

$$\dot{Q}_{cond,sphere} = \frac{T_1 - T_2}{R_{sph}}$$

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{outer radius - inner radius}}{4\pi(\text{outer radius})(\text{inner radius})(\text{thermal conductivity})}$$

Resistance Network



$R_{total} = R_{conv,1} + R_{cyl} + R_{conv,2}$

cylindrical

$$R_{total} = R_{conv,1} + R_{cond} + R_{conv,2}$$

$$= \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2}$$

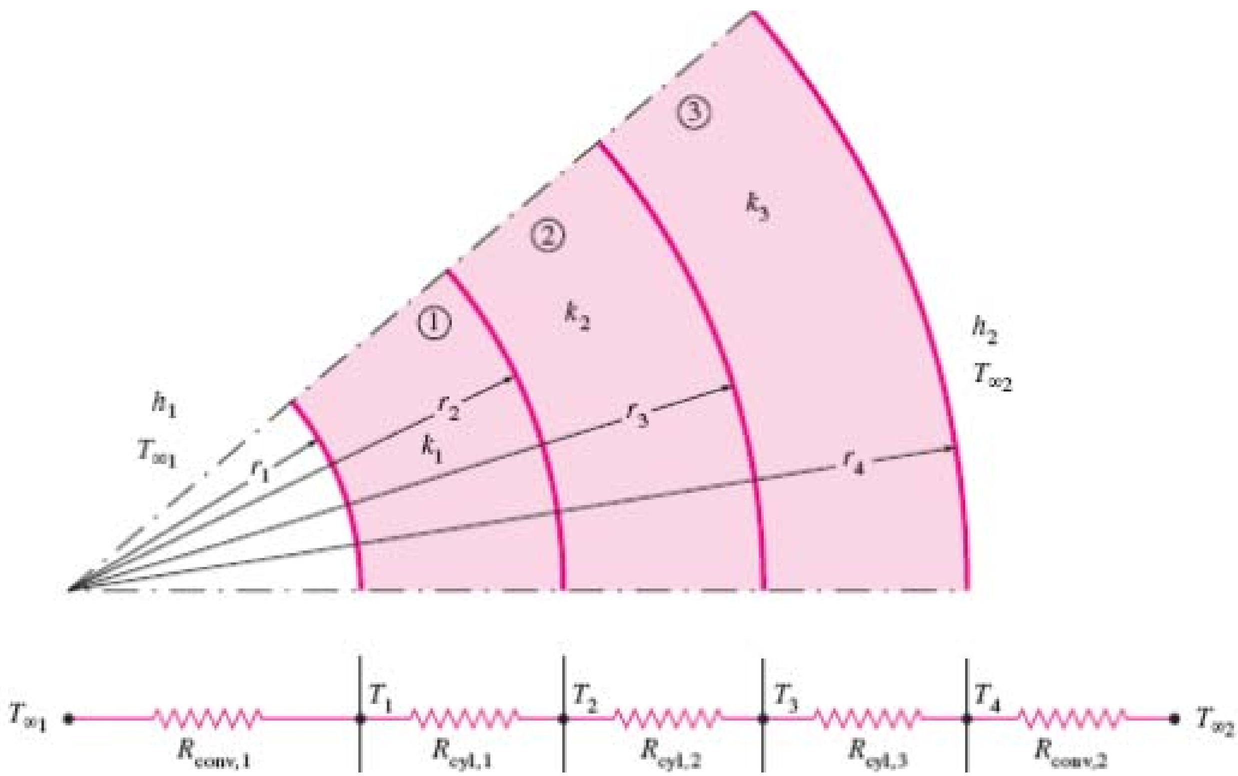
spherical

$$R_{total} = R_{conv,1} + R_{sph} + R_{conv,2}$$

$$= \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2}$$

The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

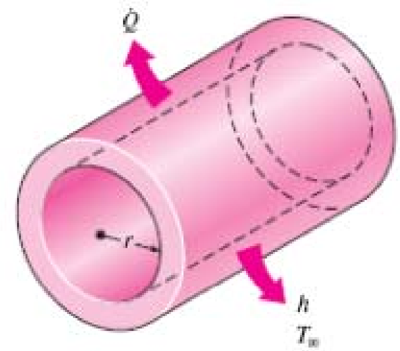
Multilayered cylinder



$$R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{cyl,3} + R_{conv,2}$$

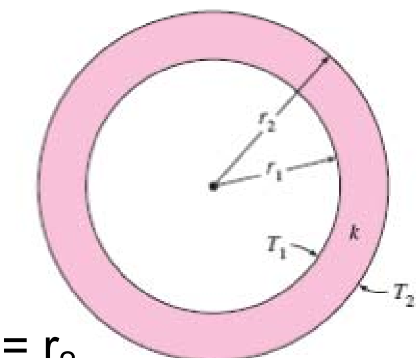
$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4}$$

Radial heat conduction through cylindrical systems



$$\frac{1}{r}\frac{\partial}{\partial r}\left(k.r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k.r\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k.\frac{\partial T}{\partial z}\right)+\dot{g}=\rho C\frac{\partial T}{\partial t}$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$



Integrating the above equation twice, $T=C_1 \ln r + C_2$

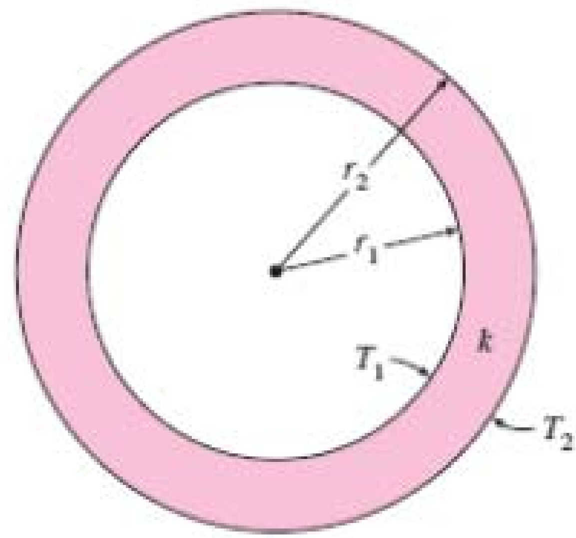
Subject to the boundary conditions, $T=T_1$ at $r = r_1$ and $T=T_2$ at $r = r_2$

$$T = \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}\ln(r) + \frac{T_1 \ln r_2 - T_2 \ln r_1}{\ln\left(\frac{r_2}{r_1}\right)}$$

$$Q = -kA_r \frac{dT}{dr}\Big|_{r=r_1} = -k.2\pi r_1 L.\frac{C_1}{r_1}$$

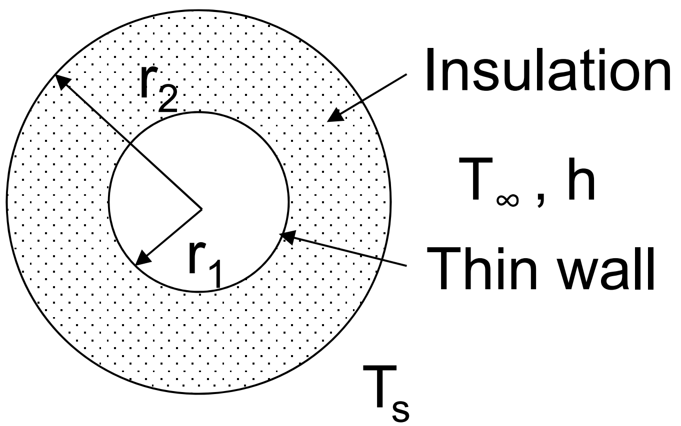
$$Q = -k.2\pi r_1 L.(T_2 - T_1).\frac{1}{r_1 \ln\left(\frac{r_2}{r_1}\right)}$$

$$= \frac{2\pi kL(T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$



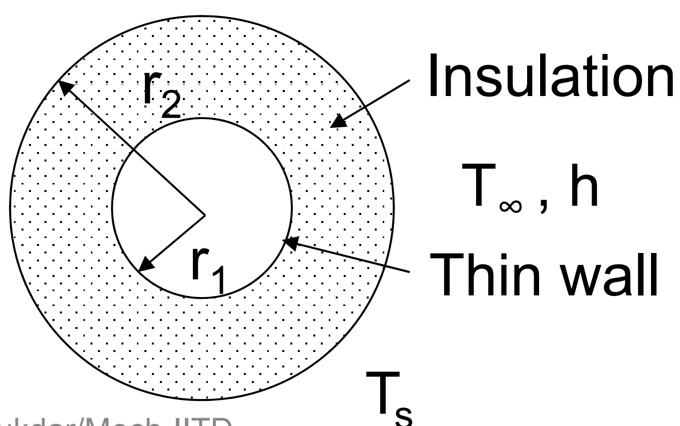
Critical Radius of Insulation

1. Steady state conditions
2. One-dimensional heat flow only in the radial direction
3. Negligible thermal resistance due to cylinder wall
4. Negligible radiation exchange between outer surface of insulation and surroundings

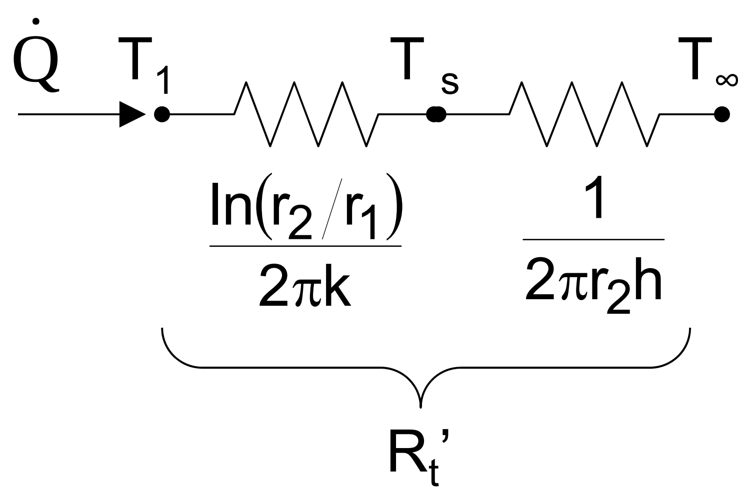


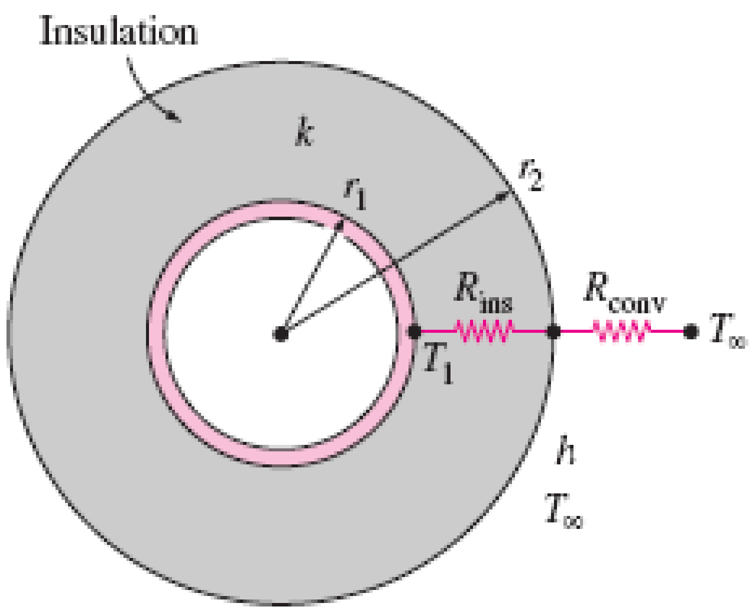
Critical Radius of Insulation

- Practically, it turns out that adding insulation in cylindrical and spherical exposed walls can initially cause the thermal resistance to decrease, thereby increasing the heat transfer rate because the outside area for convection heat transfer is getting larger. At some critical thickness, r_{cr}, the thermal resistance increases again and consequently the heat transfer is reduced.
- To find an expression for r_{cr}, consider the thermal circuit below for an insulated cylindrical wall with thermal conductivity k:

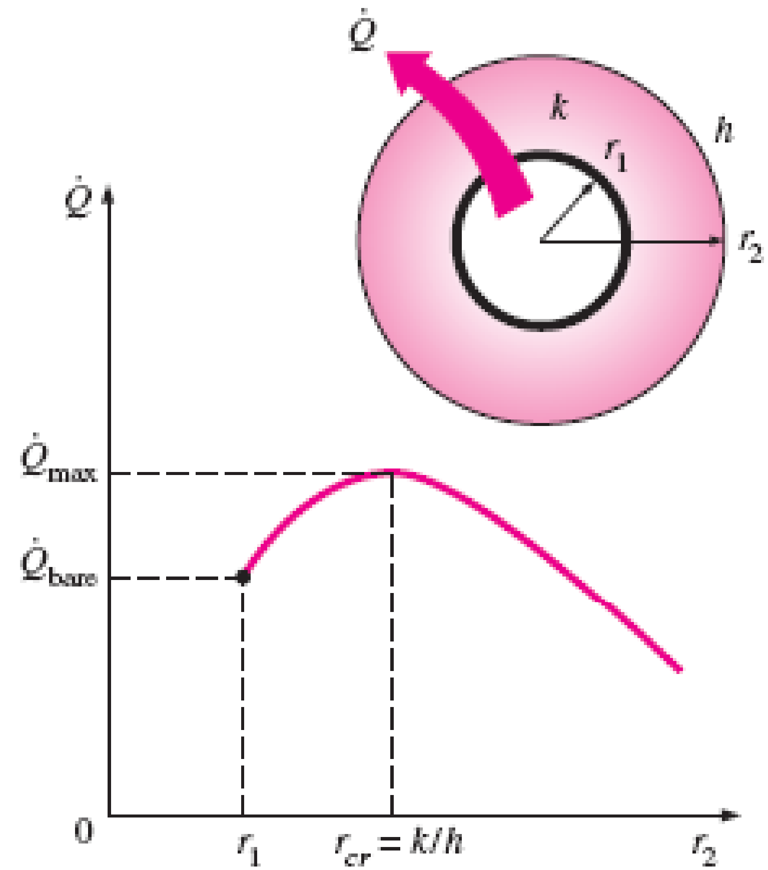


PTalukdar/Mech-IITD





An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.



- To find r_{cr}, set the overall thermal resistance $dR_t'/dr = 0$ and solve for r:

$$R_t' = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}$$

r_i = inner radius

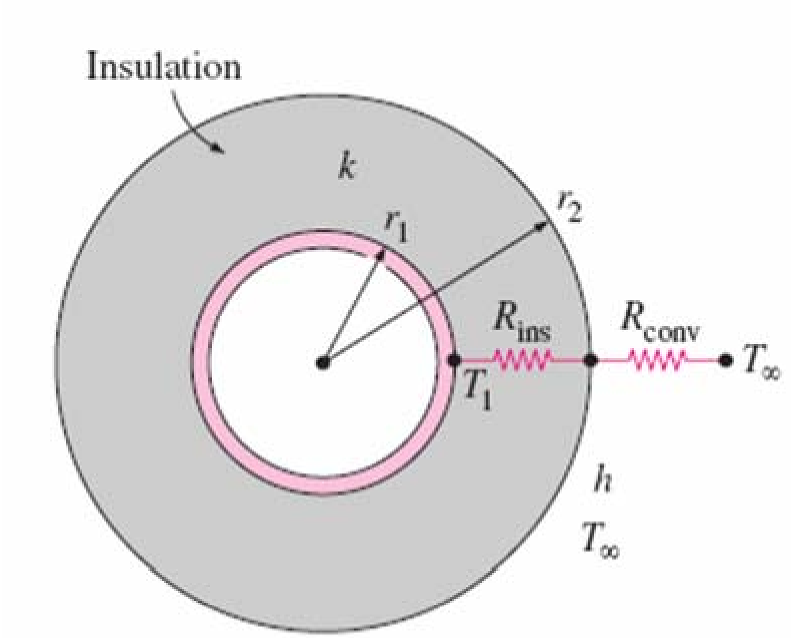
$$\frac{dR_t'}{dr} = \frac{1}{2\pi k r} - \frac{1}{2\pi r^2 h} = 0$$

$$r = r_{cr} = \frac{k}{h}$$

Similarly for a sphere

$$r_{cr} = \frac{2k}{h}$$

- For insulation thickness less that r_{cr} the heat loss increases with increasing r and for insulation thickness greater that r_{cr} the heat loss decreases with increasing r
- If k = 0.03 W/(m·K) and h = 10 W/(m²·K):
  - cylinder $r_{cr} = \frac{k}{h} = \frac{0.03 W/(m \cdot K)}{10 W/(m^2 \cdot K)} = 0.003 m = 3 mm$
  - sphere $r_{cr} = \frac{2k}{h} = 6 mm$



of r_1, h and k are constant

Total thermal resistance per unit length

$$R_{total} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k} + \frac{1}{2\pi r_2 h}$$

Heat transfer per unit length $$\frac{\dot{Q}}{L} = \frac{T_\infty - T_i}{R_{total}}$$

Optimum thickness is associated with r_2, $$\frac{dR_{total}}{dr_2} = 0$$

$$\frac{1}{2\pi k r_2} - \frac{1}{2\pi r_2^2 h} = 0 \qquad r_2 = \frac{k}{h}$$

To see the condition maximizes or minimizes the total resistance

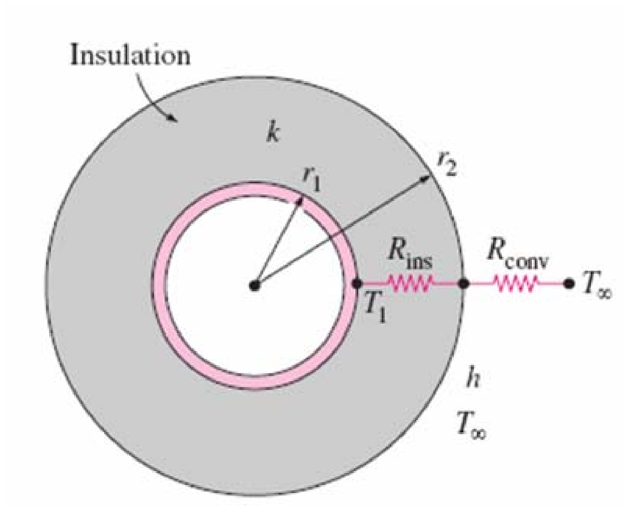
$$\frac{d^2 R_{total}}{dr_2^{\,2}} = -\frac{1}{2\pi k r_2^2} + \frac{1}{\pi r_2^3 h}$$

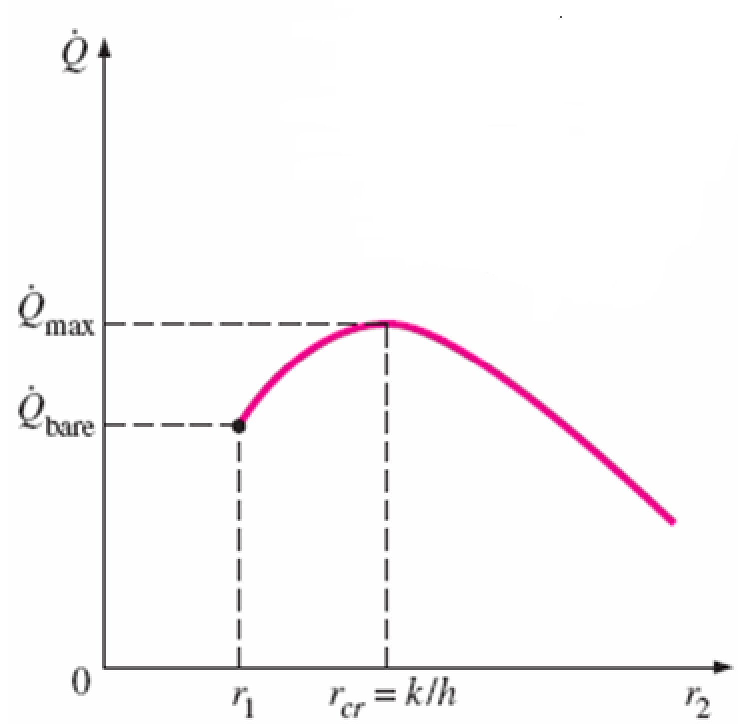
At r_2=k/h

$$\frac{d^2 R_{total}}{dr_2^{\,2}} = \frac{1}{\pi (k/h)^2}\left(\frac{1}{k} - \frac{1}{2k}\right) = \frac{1}{2\pi k^3/h^2} > 0$$

Always positive, total resistance at *k/h* is minimum

$$r_{cr,cylinder} = \frac{k}{h} \qquad \text{(m)}$$





$$r_{cr,max} = \frac{k_{max,\ insulation}}{h_{min}} \approx \frac{0.05\left(W/m\,^oC\right)}{5\left(W/m^{2\,o}C\right)} = 0.01m = 1cm$$

We can insulate hot water pipes and steam lines without worrying the critical radius of insulation

Insulation of electric wires:

-Radius of electric wires may be smaller than the critical radius

-Addition of insulation material increases heat transfer

Critical radius of insulation for spherical shell: $r_{cr,sphere} = \frac{2k}{h}$

Summary

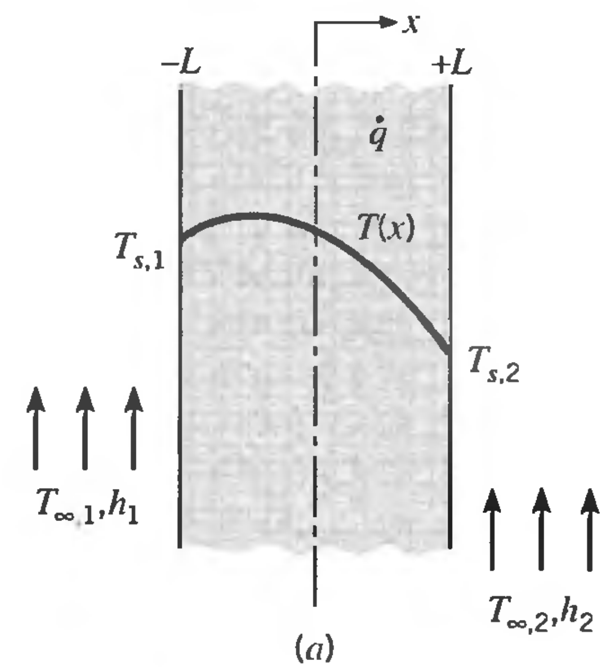
TABLE 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,cond}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

1D Conduction with Heat Generation

The Plane Wall



$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Boundary conditions:

$$T(-L) = T_{s,1}$$

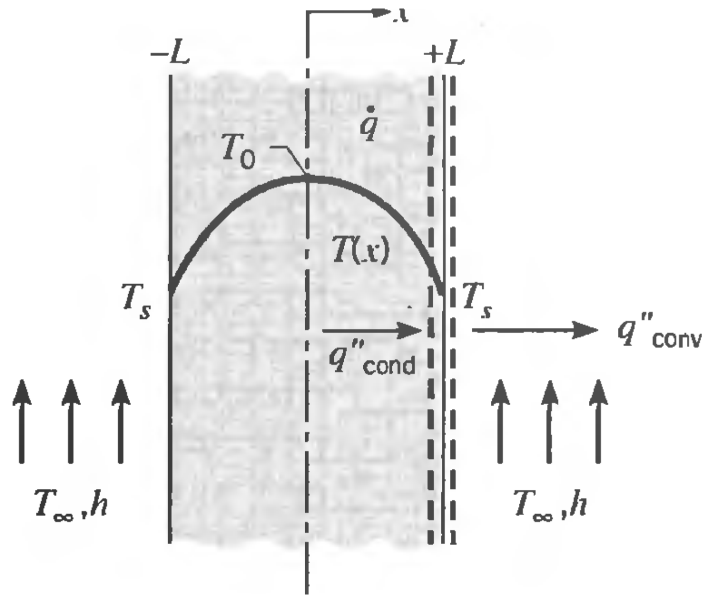
$$T(L) = T_{s,2}$$

$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L}$$

$$C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,2} + T_{s,1}}{2}$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

$$T_{s,1} = T_{s,2} \equiv T_s$$



$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

The maximum temperature exists at the midplane

$$T(0) \equiv T_o = \frac{\dot{q}L^2}{2k} + T_s$$ Put x = 0

If the surface temperature of the heat generating body is unknown and the surrounding fluid temperature is T_∞

Using energy balance $$-k\frac{dT}{dx}\Big|_{x=L} = h(T_s - T_\infty)$$

Find temperature gradient from the above Eq. at x = L

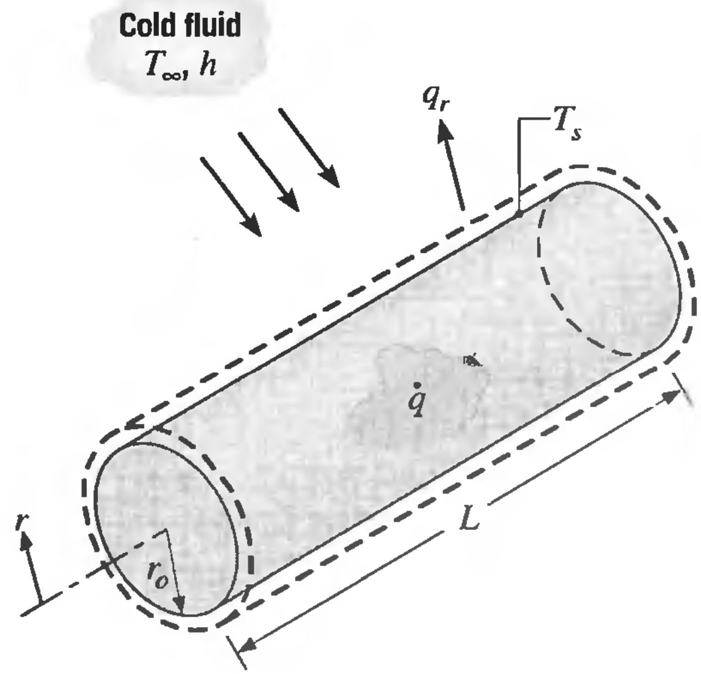
We can obtain the surface temperature $$T_s = T_\infty + \frac{\dot{q}L}{h}$$

Radial Systems

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

$$r\frac{dT}{dr}=-\frac{\dot{q}\,r^2}{2k}+C_1$$

$$T(r)=-\frac{\dot{q}\,r^2}{4k}+C_1\ln r+C_2$$



Conduction in a solid cylinder with uniform heat generation.

Boundary conditions:

$$\frac{dT}{dr}\Big|_{r=0}=0 \qquad T(r_o)=T_s$$

$$C_1=0 \qquad C_2=T_s+\frac{\dot{q}\,r_o^2}{4k}$$

$$T(r)=\frac{\dot{q}\,r_o^2}{4k}\left(1-\frac{r^2}{r_o^2}\right)+T_s$$

$$\dot{q}\left(\Pi\,r_o^2L\right)=h(2\Pi\,r_oL)(T_s-T_\infty)$$

$$T_s=T_\infty+\frac{\dot{q}\,r_o}{2h}$$